Shock-wave formation in Rosenau's extended hydrodynamics

Carlos Escudero
Departamento de Física Fundamental, Universidad Nacional de Educación a Distancia, C/Senda del Rey 9, 28040 Madrid, Spain
(Received 31 December 2004; published 20 April 2005)

We study the extended hydrodynamics proposed by Rosenau [Phys. Rev. A **40**, 7193 (1989)] in the context of a regularization of the Chapman-Enskog expansion. We are able to prove that shock waves appear in finite time in Rosenau's extended Burgers equation, and we discuss the physical implications of this fact and its connection with a possible extension of hydrodynamics to the short-wavelength domain.

DOI: 10.1103/PhysRevE.71.047302 PACS number(s): 47.40.Nm, 05.20.Dd, 05.45.−a, 02.30.Jr

The Boltzmann equation is one of the most fundamental equations in nonequilibrium statistical mechanics. This equation describes the dynamics of a rarefied gas, taking into account two basic processes: the free flight of the particles and their collisions. Due to the difficulties that a direct treatment of this equation implies, a reduced description of the Boltzmann equation is one of the major problems in kinetic theory. The equations of hydrodynamics constitute a closed set of equations for the three hydrodynamic fields: local density, local velocity, and local temperature. These equations can be derived from the Boltzmann equation by performing the Chapman-Enskog expansion [1]. This expansion is a power series expansion in the Knudsen number, which is the ratio of the free mean path between the macroscopic length. The first order of the expansion yields Euler equations, while the second order yields Navier-Stokes equations which in the case of an incompressible fluid read

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = - \nabla p + \mu \nabla^2 \mathbf{v},$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

where μ represents the viscosity of the fluid and is of the order of the Knudsen number. The next order in the Chapman-Enskog expansion yields the Burnett equations of hydrodynamics which are, unfortunately, invalid. To see this more clearly consider the viscous part of the Chapman-Enskog expansion:

$$\epsilon(\mu_0 \nabla^2 \mathbf{v} + \epsilon^2 \mu_1 \nabla^4 \mathbf{v} + \cdots), \tag{2}$$

where $\mu = \epsilon \mu_0$ and ϵ is the Knudsen number. Burnett order implies the presence of a biharmonic term proportional to ∇^4, which causes an unphysical increase in the number of boundary conditions and rends the equilibrium unstable, among other undesirable effects. While the Navier-Stokes equations give very accurate results in many domains, they usually fail when applied to predict the short-wavelength properties of the fluid, like, for instance, the propagation of ultrasounds within the fluid. This makes it very useful to develop a higher-order description of the fluid, while the Burnett order has proven itself less accurate than the Navier-Stokes order. This problem was partially solved by Rosenau in his influencing article of 1989 [2]. The idea was to regularize the Chapman-Enskog expansion using a very original comparison. First consider the power series expansion

$$\frac{1}{1-z} = 1 + z^2 + z^4 + \cdots, \tag{3}$$

where z is a complex number the modulus of which fullfills $|z| < 1$. Assuming that ϵ is small enough and taking into account the power series (3) suggests that we can recast expansion (2) into the form

$$\frac{\mu \nabla^2}{1 - \epsilon^2 m^2 \nabla^2} \mathbf{v}, \tag{4}$$

where $m^2 = \mu_1 / \mu_0$, and this operator is to be interpreted in the Fourier transform sense:

$$\left(\frac{\mu \nabla^2}{1 - \epsilon^2 m^2 \nabla^2} \mathbf{v} \right)^{\hat{}} = \frac{-\mu k^2}{1 + \epsilon^2 m^2 k^2} \hat{\mathbf{v}}. \tag{5}$$

This idea was originally proposed in the context of random walk theory [3] and has been used within this context in later works [4].

While this regularization of the Chapman-Enskog expansion seems to be a proper extension of hydrodynamics in the linear regime [2], its effect on the full nonlinear hydrodynamics is not so clear. This is due to the analytical difficulties that a mathematical treatment of the Navier-Stokes equations imply. However, it is useful to study some toy models to win a deeper understanding of hydrodynamics; to this end was developed a one-dimensional model for hydrodynamics: the Burgers equation

$$\partial_t u + u \partial_x u = \mu \partial_x^2 u. \tag{6}$$

In the same spirit, Rosenau considered a regularized Burgers equation, arguing that an understanding of this model would clarify the effect of the regularization of the Chapman-Enskog expansion on the nonlinear hydrodynamics. The rest of this work is devoted to prove the appearence of shock waves in finite time in Rosenau's regularized Burgers equation and to analyze the physical implications of this fact.

Rosenau's extended Burgers equation reads

$$\partial_t u + u \partial_x u = \mu \frac{\partial_x^2}{1 - \epsilon^2 \partial_x^2} u, \tag{7}$$

where we have set, without loss of generality, $\mu_1 / \mu_0 = 1$. To prove shock-wave formation we will exploit the analogy between the viscous Burgers equation (the inviscid Burgers equation is obtained just by setting $\mu=0$) and the Keller-Segel system [5]:

$$\partial_t v = \mu \partial_x^2 v + \partial_x(v \partial_x w), \tag{8}$$

$$\partial_x^2 w = -v. \tag{9}$$

Note that we recover the viscous Burgers equation performing the substitution $u=\partial_x w$ in the system (8) and (9). Consider now the following modified Keller-Segel system:

$$\partial_t v = \mu \frac{\partial_x^2}{1-\epsilon^2 \partial_x^2} v + \partial_x(v \partial_x w), \tag{10}$$

$$\partial_x^2 w = -v. \tag{11}$$

We can recover Rosenau's extended Burgers equation by performing again the substitution $u=\partial_x w$ in this last system. We will consider homogeneous Dirichlet boundary conditions $v|_{\partial\Omega}=w|_{\partial\Omega}=0$, where Ω is the closed interval $\Omega=[-L,L]$. From the system (10) and (11) we get

$$\frac{1}{2}\frac{d}{dt}\|v(\cdot,t)\|_{L^2(\Omega)}^2 = \int_\Omega v v_t dx = \mu \int_\Omega v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v dx - \int_\Omega v \partial_x w \partial_x v dx + \int_\Omega v^3 dx. \tag{12}$$

Now, we are going to estimate all the terms appearing on the right-hand side of this equation.

Integrating by parts the second term on the right-hand side of Eq. (12),

$$\int_\Omega v \partial_x w \partial_x v dx = v^2 \partial_x w|_{\partial\Omega} - \int_\Omega \partial_x v \partial_x w v dx - \int_\Omega v \partial_x^2 w v dx, \tag{13}$$

which implies

$$\int_\Omega v \partial_x w \partial_x v dx = -\frac{1}{2}\int_\Omega v^2 \partial_x^2 w dx = \frac{1}{2}\int_\Omega v^3 dx. \tag{14}$$

The first term on the right-hand side of Eq. (12) can be estimated as follows:

$$\int_\Omega v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v dx \leqslant \left| \int_\Omega v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v dx \right| \leqslant \int_\Omega \left| v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v \right| dx \leqslant \|v\|_{L^2(\Omega)} \times \left\| \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v \right\|_{L^2(\Omega)}, \tag{15}$$

where we have used Hölder's inequality (see below). By performing the shift of variables $y=x/\epsilon$, we get

$$\left\| \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v \right\|_{L^2(\Omega)} = \frac{1}{\epsilon^{(3/2)}} \left\| \frac{\partial_y^2}{1-\partial_y^2} v \right\|_{L^2(\Omega/\epsilon)} \leqslant \frac{N}{\epsilon^{(3/2)}} \|v\|_{L^2(\Omega/\epsilon)}, \tag{16}$$

where $N=|\partial_y^2(1-\partial_y^2)^{-1}|$. Let us clarify a bit this last step. We have used the fact that the operator $\nabla^2(1-\nabla^2)^{-1}$ is bounded on every L^p space, with $1\leqslant p\leqslant\infty$. This means that we can assure that $\|\nabla^2(1-\nabla^2)^{-1}f\|_{L^p(\Omega)}\leqslant N\|f\|_{L^p(\Omega)}$ for every f belonging to $L^p(\Omega)$ and a constant N that does not depend on f (and thus N is called the norm of the operator). This fact can be easily seen once one realizes that the Fourier transform of the operator $\nabla^2(1-\nabla^2)^{-1}$ is a bounded function of the wave vector, and a rigorous proof can be found in Ref. [6]. We can again shift variables $x=\epsilon y$ to get

$$\int_\Omega v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v dx \leqslant \frac{N}{\epsilon^2}\|v\|_{L^2(\Omega)}^2. \tag{17}$$

Finally, we can conclude our estimate as follows:

$$\int_\Omega v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v dx \geqslant -\left| \int_\Omega v \frac{\partial_x^2}{1-\epsilon^2\partial_x^2} v dx \right| \geqslant -\frac{N}{\epsilon^2}\|v\|_{L^2(\Omega)}^2. \tag{18}$$

Now we are going to estimate the third term in Eq. (12):

$$\int_\Omega v^3 dx = \|v\|_{L^3(\Omega)}^3. \tag{19}$$

Hölder's inequality reads (for a rigorous proof of Hölder's inequality see [7])

$$\int_\Omega |fg| dx \leqslant \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)},$$

$$1\leqslant p,q\leqslant\infty, \quad \frac{1}{p}+\frac{1}{q}=1. \tag{20}$$

Choosing $g=1$ we get

$$\int_\Omega |f| dx \leqslant C\|f\|_{L^p(\Omega)}, \tag{21}$$

where $C=|\Omega|^{1/q}$. With this estimate we can claim that

$$\|v\|_{L^2(\Omega)}^2 = \int_\Omega v^2 dx \leqslant C\|v^2\|_{L^p(\Omega)} = C\left(\int_\Omega v^{2p} dx\right)^{(1/p)} = C\left(\int_\Omega v^3 dx\right)^{(2/3)} = C\|v\|_{L^3(\Omega)}^2, \tag{22}$$

where we have chosen $p=3/2$ (and correspondingly $q=3$). This implies that

$$\|v\|_{L^3(\Omega)} \geqslant D\|v\|_{L^2(\Omega)}, \tag{23}$$

where $D=|\Omega|^{-1/6}$. Therefore, we have the final estimate

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \geqslant A(\|v\|^2_{L^2(\Omega)})^{(3/2)} - B\|v\|^2_{L^2(\Omega)}, \tag{24}$$

where $A,B>0$ are constants, $A=|\Omega|^{-1/2}$, and $B=2N\mu/\epsilon^2$. We are thus going to study the dynamical system

$$\frac{dx}{dt} = Ax^{3/2} - Bx. \tag{25}$$

This system has two fixed points $x=0$ and $x=(B/A)^2>0$. A linear stability analysis reveals that the positive fixed point is linearly unstable, meaning that every initial condition $x_0>(B/A)^2$ will stay above this value for all times. Further, we know that the solution will grow without bound in this case, so we can claim the existence of two constants $t_0<\infty$ and $0<C_0<A$, such that $Ax^{3/2}(t)-Bx(t)>C_0x^{3/2}(t)$ for every $t>t_0$. This implies that

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} > C_0(\|v\|^2_{L^2(\Omega)})^{(3/2)} \tag{26}$$

for $t>t_0$ and for an adequate initial condition. Solving this equation gives:

$$\|v(\cdot,t)\|^2_{L^2(\Omega)} > \frac{1}{\sqrt{\|v(\cdot,t_1)\|^{-1}_{L^2(\Omega)} - \frac{C_0}{2}t}} \tag{27}$$

for $t>t_1>t_0$ and for an adequate initial condition. And every adequate initial condition must fullfill

$$\|v(\cdot,0)\|^2_{L^2(\Omega)} > \frac{4N^2\mu^2}{\epsilon^4}|\Omega| + \frac{4N\mu}{\epsilon^2}\|v(\cdot,0)\|_{L^1(\Omega)} + \frac{1}{|\Omega|}\|v(\cdot,0)\|^2_{L^1(\Omega)}, \tag{28}$$

like, for instance, $v(x,0)=(x^2+\delta)^{-1/4}-(L^2+\delta)^{-1/4}$ and δ small enough. Thus we are finally led to conclude that the system does blow up in finite time. If we recover $v=-\partial_x u$, we see that the first spatial derivative of u becomes singular in finite time. This means that the solution to Eq. (7) develops a shock wave in finite time (or what is the same, a discontinuity in the flow appears), in contrast to the viscous Burgers equation and analogously to the inviscid Burgers equation $\partial_t v=-v\partial_x v$ [7]. The inviscid Burgers equation is a one-dimensional model for the Euler equations, while the viscous Burgers equation simulates the Navier-Stokes equations. This suggests that the regularizating procedure implies a return to a lower order in the Chapman-Enskog expansion.

It was already argued by Rosenau that this kind of regularization of the Chapman-Enskog expansion was only valid in the linear regime, while nonlinear terms might be present in the full nonlinear hydrodynamics. These terms are expected to have a deep impact on the dynamics of the fluid, the reason being as follows. Whether or not the Navier-Stokes equations become singular in finite time is still unkown and it is actually one of the most important open problems in mathematics. What one would expect from a physical point of view is that these possible divergences smooth out if we look closer to the fluid, taking into account higher-order terms in a (complete) regularized Chapman-Enskog expansion. What we have seen in this work is that the linear regularized theory is able to convert a smooth solution into a singular one, so one would expect that a regularized Navier-Stokes equation of the form

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mu\frac{\nabla^2}{1-m^2\epsilon^2\nabla^2}\mathbf{v},$$

$$\nabla\cdot\mathbf{v} = 0 \tag{29}$$

is less regular than the original Navier-Stokes equation. We expect the presence of these nonlinear terms to regularize enough this equation that one would be able to prove global existence in time of the solution and this way give a precise physical meaning to the possible divergences arising in the original Navier-Stokes equation.

The author gratefully acknowledges illuminating discussions with Antonio Córdoba, Diego Córdoba, and Francisco Gancedo. This work has been partially supported by the Ministerio de Ciencia y Tecnología (Spain) through Project No. BFM2001-0291 and by UNED.

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1970).
[2] P. Rosenau, Phys. Rev. A **40**, 7193 (1989).
[3] C. R. Doering, P. S. Hagan, and P. Rosenau, Phys. Rev. A **36**, 985 (1987).
[4] P. G. Kevrekidis, I. G. Kevrekidis, A. R. Bishop, and E. S. Titi, Phys. Rev. E **65**, 046613 (2002).
[5] E. F. Keller and L. A. Segel, J. Theor. Biol. **26**, 399 (1970).
[6] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, 1970).
[7] L. C. Evans, *Partial Differential Equations* (American Mathematical Society, Providence, RI, 2002).